– EN KÜÇÜK KARELER YÖNTEMİ –

- Eğri uydurmanın etkin yöntemlerindendir.
- Enterpolasyona göre iki önemli avantajı vardır.
  - ↳ Enterpolasyonla elde edilen fonksiyon, gerçek fonksiyonu belli aralıkta temsil eder.
  - ↳ Enterpolasyonlarla elde edilen polinom fonksiyonunun kuvveti, gerekli değer sayısından bir eksiktir. Dolayısıyla, uygulamadan elde edilecek çok sayıda değerden, yüksek dereceli polinomlar elde etmek gerekecek ve bu da hesaplamaları hem zorlaştıracak hem de uzatacaktır.

- En küçük kareler yöntemi; uydurulan yaklaşık fonksiyon değerleri ile gerçek fks. değerleri arasındaki farkları kareleri toplamının minimum yapılması esasına dayanan en etkili eğri uydurma yöntemidir.

$y=f(x)$ → gerçek fks. $z=\rho(x)$ → uydurulan fks.

bilinen n tane nokta değerinden;

$$\sum_{i=1}^{n}\left[\rho(x_i)-f(x_i)\right]^2$$ 'nin minimum yapılmasıyla

$z=\rho(x)$ fks. katsayıları elde edilir.

- Farkların yani hatanın minimum olması fark fonk. birinci türevinin sıfıra eşitlenmesiyle sağlanır.

Ör; Birinci dereceden polinom uydurma

n tane nokta değeri (x_i, y_i) için $\rho(x)=a_1+a_2x$ polinomunu elde ediniz.

Hata fks.

$$H(a_1,a_2)=\left[a_1+a_2 \cdot x_i-y_i\right]^2$$

– H'nin sıfır olması için $\dfrac{\partial H(a_1,a_2)}{\partial a_i}=0$ olmalıdır.

$$\frac{\partial H(a_1,a_2)}{\partial a_i}=0 \ ; \ i=1,2$$

a_1, a_2'ye göre türevler:

$$\frac{\partial H}{\partial a_1}=2\cdot\sum_{i=1}^{n}\left[a_1+a_2\cdot x_i-y_i\right]\cdot 1=0$$

$$\frac{\partial H}{\partial a_2}=2\cdot\sum_{i=1}^{n}\left[a_1+a_2\cdot x_i-y_i\right]\cdot x_i=0$$

Bu sistem düzenlenirse:

$$\left.\begin{aligned} n\cdot a_1+a_2\cdot\sum_{i=1}^{n}x_i &= \sum_{i=1}^{n}y_i \\ a_1\cdot\sum_{i=1}^{n}x_i+a_2\cdot\sum_{i=1}^{n}x_i^2 &= \sum_{i=1}^{n}x_i y_i \end{aligned}\right\} \text{Cramer yöntemi ile çözelim}$$

$$\begin{bmatrix} n & \sum_{i=1}^{n}x_i \\ \sum_{i=1}^{n}x_i & \sum_{i=1}^{n}x_i^2 \end{bmatrix}\cdot\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}=\begin{bmatrix} \sum_{i=1}^{n}y_i \\ \sum_{i=1}^{n}x_i\cdot y_i \end{bmatrix}$$

$$\Delta=\begin{vmatrix} n & \sum_{i=1}^{n}x_i \\ \sum_{i=1}^{n}x_i & \sum_{i=1}^{n}x_i^2 \end{vmatrix}, \quad \Delta_1=\begin{vmatrix} \sum_{i=1}^{n}y_i & \sum_{i=1}^{n}x_i \\ \sum_{i=1}^{n}x_i\cdot y_i & \sum_{i=1}^{n}x_i^2 \end{vmatrix}$$

$$\Delta_2=\begin{vmatrix} n & \sum_{i=1}^{n}y_i \\ \sum_{i=1}^{n}x_i & \sum_{i=1}^{n}x_i\cdot y_i \end{vmatrix}$$

$$a_1=\frac{\Delta_1}{\Delta}$$

$$a_2=\frac{\Delta_2}{\Delta}$$

$$X1=\sum_{i=1}^{n} x_i\ ,\quad X2=\sum_{i=1}^{n} x_i^2\ ,\quad XY=\sum_{i=1}^{n} x_i . y_i$$

$$Y1=\sum_{i=1}^{n} y_i \implies$$

$$a_1=\frac{Y1.X2-X1.XY}{n.X2-X1^2}\ ,\quad a_2=\frac{n.XY-X1.Y1}{n.X2-X1^2}$$

Ör:

x	0	1	2	4	7
y	1	4	7	13	22

$\rho(x)=a_1+a_2.x$ → yaklaşımını uydurun.

$$\rho(x_i)=a_1+a_2.x_i$$

$$H(a_1,a_2)=\sum_{i=1}^{n}\left[\rho(x_i)-y_i\right]^2$$

$$\frac{\partial H}{\partial a_1}=0 \quad \text{ve} \quad \frac{\partial H}{\partial a_2}=0$$

$$\frac{\partial \sum_{i=1}^{n}\left[a_1+a_2.x_i-y_i\right]^2}{\partial a_1}=0$$

$$2.\sum_{i=1}^{n}\left[a_1+a_2.x_i-y_i\right].1=0$$

$$n.a_1+a_2.\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}y_i=0$$

$$\frac{\partial \sum_{i=1}^{n}\left[a_1+a_2.x_i-y_i\right]^2}{\partial a_2}=0$$

$$2 \cdot \sum_{i=1}^{n} \left[a_1 + a_2 \cdot x_i - y_i\right] \cdot x_i = 0$$

$$a_1 \cdot \sum_{i=1}^{n} x_i + a_2 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i \cdot y_i$$

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i \cdot y_i \end{bmatrix}$$

$$\Delta = \begin{vmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} \sum_{i=1}^{n} y_i & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} x_i^2 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} n & \sum_{x=1}^{n} y_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i \cdot y_i \end{vmatrix} \qquad a_1 = \frac{\Delta_1}{\Delta}, \quad a_2 = \frac{\Delta_2}{\Delta}$$

$$\Delta = \begin{vmatrix} 5 & 14 \\ 14 & 70 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} 47 & 14 \\ 224 & 70 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} 5 & 47 \\ 14 & 224 \end{vmatrix}$$

$$\Delta = 154 \qquad \Delta_1 = 154 \qquad \Delta_2 = 462$$

$$a_1 = \frac{\Delta_1}{\Delta} = 1 \qquad a_2 = \frac{\Delta_2}{\Delta} = 3$$

$$\boxed{g(x) = 1 + 3x}$$

— İkinci Dereceden Polinom Uydurma —

— n tane nokta değeri (x_i, y_i) kullanarak $p(x)=a_1+a_2x+a_3x^2$

$$H(a_1,a_2,a_3)=\sum_{i=1}^{n}\left[a_1+a_2x_i+a_3.x_i^2-y_i\right]^2$$

şeklindedir.

— Hata fonksiyonunun minimum olması için 1. türevinin sıfıra eşit olması gerekir.

$$\frac{\partial H(a_1,a_2,a_3)}{\partial a_i}=0 \quad , i=1,2,3$$

$$\frac{\partial H}{\partial a_1}=2.\sum_{i=1}^{n}\left[a_1+a_2.x_i+a_3.x_i^2-y_i\right].1=0$$

$$\frac{\partial H}{\partial a_2}=2.\sum_{i=1}^{n}\left[a_1+a_2.x_i+a_3.x_i^2-y_i\right].x_i=0$$

$$\frac{\partial H}{\partial a_3}=2.\sum_{i=1}^{n}\left[a_1+a_2.x_i+a_3.x_i^2-y_i\right].x_i^2=0$$

$$n.a_1+a_2.\sum_{i=1}^{n}x_i+a_3.\sum_{i=1}^{n}x_i^2=\sum_{i=1}^{n}y_i$$

$$a_1.\sum_{i=1}^{n}x_i+a_2\sum_{i=1}^{n}x_i^2+a_3\sum_{i=1}^{n}x_i^3=\sum_{i=1}^{n}y_i.x_i$$

$$a_1.\sum_{i=1}^{n}x_i^2+a_2.\sum_{i=1}^{n}x_i^3+a_3\sum_{i=1}^{n}x_i^4=\sum_{i=1}^{n}y_i.x_i^2$$

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} y_i . x_i \\ \sum_{i=1}^{n} y_i . x_i^2 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} \sum_{i=1}^{n} y_i & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} y_i . x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} y_i . x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} n & \sum_{i=1}^{n} y_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i . x_i & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} y_i . x_i^2 & \sum_{i=1}^{n} x_i^4 \end{vmatrix}, \quad \Delta 3 = \begin{vmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} y_i . x_i \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} y_i . x_i^2 \end{vmatrix}$$

$$a_1 = \frac{\Delta 1}{\Delta}, \quad a_2 = \frac{\Delta 2}{\Delta}, \quad a_3 = \frac{\Delta 3}{\Delta}$$

$$X1 = \sum_{i=1}^{n} x_i, \quad X2 = \sum_{i=1}^{n} x_i^2, \quad X3 = \sum_{i=1}^{n} x_i^3, \quad X4 = \sum_{i=1}^{n} x_i^4$$

$$XY = \sum_{i=1}^{n} x_i . y_i, \quad X2Y = \sum_{i=1}^{n} x_i^2 . y_i, \quad Y1 = \sum_{i=1}^{n} y_i$$

ise,

$$\Delta = n.X2.X4 + 2X1.X2.X3 - X2^3 - n.X3^2 - X4.X1^2$$

$$\Delta_1 = X2.X4.Y1 + X1.X3\,X2Y + X2.X3.XY - X2Y.X2^2 - Y1.X3^2 - X1.X4.XY$$

$$\Delta_2 = n.X4.XY + X2.X3.Y1 + X1.X2.X2Y - XY.X2^2 - n.X3.X2Y - X1.X4.Y1$$

$$\Delta_3 = n.X2.X2Y + X1.X2.XY + X1.X3.Y1 - Y1.X2^2 - n.X3.XY - X2Y.X1^2$$

Ör:

x	0	2	3	5	8
y	-6	0	6	24	66

noktaları için en uygun $p(x) = a_1 + a_2.x + a_3.x^2$ ikinci derece polinomunu uydurunuz.

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} y_i.x_i \\ \sum_{i=1}^{n} y_i.x_i^2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 18 & 102 \\ 18 & 102 & 672 \\ 102 & 672 & 4818 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 90 \\ 666 \\ 4878 \end{bmatrix}$$

$a_1 = -6$, $a_2 = 1$, $a_3 = 1$ $\boxed{y = -6 + x + x^2}$

— Yüksek Dereceden Polinom Uydurma —

— n tane (x_i, y_i) nokta değerinden

$$p(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \dots + a_p x^{p-1}$$

için;

$$\frac{\partial H(a_1, a_2, \dots a_p)}{\partial a_i} = 0 \quad , i = 1, 2, 3, \dots, p$$

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i & \cdots & \sum_{i=1}^{n} x_i^{p} \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^{2} & & \sum_{i=1}^{n} x_i^{p+1} \\ \sum_{i=1}^{n} x_i^{2} & \sum_{i=1}^{n} x_i^{3} & & \sum_{i=1}^{n} x_i^{p+2} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{n} x_i^{p} & \sum_{i=1}^{n} x^{p+1} & \cdots & \sum_{i=1}^{n} x_i^{2p} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} y_i . x_i \\ \sum_{i=1}^{n} y_i . x_i^{2} \\ \vdots \\ \sum_{i=1}^{n} y_i . x_i^{p} \end{bmatrix}$$

— Üstel Fonksiyon Uydurma —

— n tane ~~nokta~~ değeri (x_i, y_i) kullanarak $\rho(x) = a_1 . e^{a_2 x}$ şeklinde bir üstel fonksiyon elde etmek için öncelikle bu fks. doğrusallaştırılmalıdır.

— Çünkü a'lara göre kısmi türev alınıp sıfıra eşitlendiğinde lineer denklem sistemi oluşmaz.

$$\rho(x) = y = a_1 . e^{a_2 . x}$$

$$\ln(y) = \ln(a_1 . e^{a_2 . x})$$

$$= \ln(a_1) + \ln(e^{a_2 . x})$$

$$\ln(y) = \ln a_1 + a_2 . x \text{ , olur.}$$

$$H(a_1, a_2) = \sum_{i=1}^{n} [\ln(a_1) + a_2 . x_i - \ln(y_i)]^2$$

$$\frac{\partial H(a_1, a_2)}{\partial a_i} = 0 \; ; \; i = 1, 2$$

$$\frac{\partial H}{\partial a_1} = 2 . \sum [\ln a_1 + a_2 . x_i - \ln y_i] . \frac{1}{a_1} = 0$$

$$\frac{\partial H}{\partial a_2} = 2 . \sum [\ln a_1 + a_2 . x_i - \ln y_i] . x_i = 0$$

$$n \cdot \ln(a_1) + a_2 \cdot \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \ln(y_i)$$

$$\sum_{i=1}^{n} \ln(a_1) \cdot x_i + a_2 \cdot \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i \cdot \ln(y_i)$$

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} \ln(a_1) \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \ln(y_i) \\ \sum_{i=1}^{n} x_i \cdot \ln(y_i) \end{bmatrix}$$

$$X1 = \sum_{i=1}^{n} x_i, \quad X2 = \sum_{i=1}^{n} x_i^2, \quad LY = \sum_{i=1}^{n} \ln(y_i)$$

$$XLY = \sum_{i=1}^{n} [x_i \cdot \ln(y_i)]$$

$$\ln(a_1) = \frac{LY \cdot X2 - XLY \cdot X1}{n \cdot X2 - X1^2} \Rightarrow a_1 = e^{\frac{LY \cdot X2 - XLY \cdot X1}{n \cdot X2 - X1^2}}$$

$$a_2 = \frac{n \cdot XLY - LY \cdot X1}{n \cdot X2 - X1^2}.$$

Ör:

x	0	1	2	3	5	8
y	3	8,155	22,167	60,257	445,239	8942,874

$p(x) = a_1 \cdot e^{a_2 \cdot x} \rightarrow$ eğrisini uydurun.

$$X1 = \sum_{i=1}^{6} x_i = 19, \quad X2 = \sum_{i=1}^{6} x_i^2 = 103$$

$$LY = \sum_{i=1}^{6} \ln(y_i) = 25,592, \quad XLY = \sum_{i=1}^{6} [x_i \cdot \ln(y_i)] = 123,87$$

$$\left.\begin{aligned} a_1 &= e^{\frac{LY \cdot X2 - XLY \cdot X1}{nX2 - X1^2}} \cong 3 \\ a_2 &= \frac{n \cdot XLY - LY \cdot X1}{n \cdot X2 - X1^2} \cong 1 \end{aligned}\right\} p(x) = 3e^x.$$

— Trigonometrik fonksiyon Uydurma —

n tane nokta değeri $(x_i; y_i)$ kullanarak ~~g(x)~~ $g(x) = a_1 . \sin(x) + a_2 . \cos(x)$ şeklinde ~~t~~trigonometrik bir fks. uydurmak için;

$$H(a_1, a_2) = \sum_{i=1}^{n} \left[a_1 . \sin(x_i) + a_2 . \cos(x_i) - y_i \right]^2$$

$$\frac{\partial H(a_1, a_2)}{\partial a_i} = 0 \quad , \quad i = 1, 2$$

$$2 . \sum_{i=1}^{n} \left[a_1 . \sin(x_i) + a_2 . \cos(x_i) - y_i \right] . \sin(x_i) = 0$$

$$2 . \sum_{i=1}^{n} \left[a_1 . \sin(x_i) + a_2 . \cos(x_i) - y_i \right] . \cos(x_i) = 0$$

$$a_1 . \sum_{i=1}^{n} \sin^2(x_i) + a_2 . \sum_{i=1}^{n} \sin(x_i) . \cos(x_i) = \sum_{i=1}^{n} y_i . \sin(x_i)$$

$$a_1 \sum_{i=1}^{n} \sin(x_i) \cos(x_i) + a_2 . \sum_{i=1}^{n} \cos^2(x_i) = \sum_{i=1}^{n} y_i . \cos(x_i)$$

$$\begin{bmatrix} \sum_{i=1}^{n} \sin^2(x_i) & \sum_{i=1}^{n} \sin(x_i) . \cos(x_i) \\ \sum_{i=1}^{n} \sin(x_i) . \cos(x_i) & \sum_{i=1}^{n} \cos^2(x_i) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \sin(x_i) \\ \sum_{i=1}^{n} y_i . \cos(x_i) \end{bmatrix}$$

$S2=\sum_{i=1}^{n}\sin^2(x_i)$, $C2=\sum_{i=1}^{n}\cos^2(x_i)$, $CS=\sum_{i=1}^{n}[\cos(x_i).\sin(x_i)]$

$SY=\sum_{i=1}^{n}[y_i.\sin(x_i)]$, $CY=\sum_{i=1}^{n}[y_i.\cos(x_i)]$

$$\begin{bmatrix} S2 & CS \\ CS & C2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}=\begin{bmatrix} SY \\ CY \end{bmatrix}$$

$\Delta = S2.C2 - CS^2$, $\Delta 1 = SY.C2 - CY.CS$

$\Delta 2 = S2.CY - CS.SY$

$a_1=\frac{\Delta 1}{\Delta}$, $a_2=\frac{\Delta 2}{\Delta}$

Örn:

x (rad)	0	1	2	3	4	5
y	3	3,304	0,57	−2,688	−3,474	−1,067

noktaları için en uygun $p(x)=a_1.\sin(x)+a_2.\cos(x)$ şeklindeki trigonometrik fks. uydurunuz.

$S2=\sum_{i=1}^{6}\sin^2(x_i)=3,047$, $C2=\sum_{i=1}^{6}\cos^2(x_i)=2,953$

$CS=\sum_{i=1}^{6}[\cos(x_i).\sin(x_i)]=0,159$

$SY=\sum_{i=1}^{6}[y_i.\sin(x_i)]=6,572$

$CY=\sum_{i=1}^{6}[y_i.\cos(x_i)]=9,177$

$\Delta = S2.C2 - CS^2 \cong 8,972$

$\Delta 1 = SY.C2 - CY.CS \cong 17,948$

$\Delta 2 = S2.CY - CS.SY \cong 26,918$

$a_1=\frac{\Delta 1}{\Delta}\cong 2$

$a_2=\frac{\Delta 2}{\Delta}\cong 3$

$p(x)=2.\sin(x)+3.\cos(x)$